Question 5

Let V and W be finite dimensional vector spaces and $f: V \longrightarrow W$ be a linear transformation with kernel of f is the set $Ker\ (f) = \{v | f(v) = 0\}$.
Prove f is one to one if and only if $Ker(f) = \{0\}$. **(10 marks)**

Solution:

Let $f \in L(V, W)$.
$(\Leftarrow)$ Assume $Ker(f) = \{0\}$.
$f(v_1) = f(v_2) \Rightarrow f(v_1) - f(v_2) = 0$
$\Rightarrow f(v_1 - v_2) = 0$ since f is linear
$\Rightarrow v_1 - v_2 \in Ker(f)$
But then $Ker(f) = \{0\}$ by assumption, thus $v_1 - v_2 = 0$
$\Rightarrow v_1 = v_2$ $\therefore f$ is one to one
$(\Rightarrow)$ Assume f is $1-1$.
Look, $f(0) = f\big(w + (-w)\big) = f(w) - f(w) = 0$.
Now if $w \in Ker(f)$, therefore $f(w) = 0 = f(0)$
f is $1-1$ by assumption, $\therefore w = 0$. Thus $Ker(f) = \{0\}$ since w is arbitrary. ■